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STUDYING THE DARCY–STEFAN PROBLEM ON PHASE TRANSITION IN A SATURATED POROUS SOIL

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The Cauchy problem for the Darcy–Stefan model, which describes the process of freezing (thawing) of a saturated porous soil with allowance for liquid-phase filtration, is considered. The model includes the Darcy law, the equation of liquid-phase incompressibility, the equation of absence of solid-phase motion, the equation of energy balance in the porous soil–saturating continuous medium system, and also the Stefan condition and the condition of continuity of the normal components of the velocity field at the interface boundary. The existence of generalized solutions of the problem satisfying an additional condition of entropy nondecreasing in a thermomechanical system (i.e., the second law of thermodynamics) is proved by the method of the kinetic equation.

Key words: *filtration in a porous soil, Darcy law, Stefan problem, freezing, thawing, entropy, kinetic equation.*

Introduction. Mathematical modeling of soil freezing (thawing) phenomena with allowance for convection of the saturating liquid phase is necessary for scientific support of various engineering processes in industry and agriculture [1]. The majority of the models used combine the Stefan problem to describe phase transformations in a continuous medium and the Darcy law to describe the dynamics of filtration of viscous continuous media through an incompressible porous soil. A rather generic multidimensional Darcy–Stefan model was proposed in [2] to describe the processes of freezing (thawing) of a viscous liquid in a motionless porous soil with identical values of density in the liquid and frozen phases and with allowance for the buoyancy force depending nonlinearly on temperature. An initial-boundary problem was considered for this model in [2], and the existence of a weak generalized solution was proved. (The proof was performed by the classical methods of the theory of elliptic and parabolic second-order equations.) The sought functions were the filtration rate, the pressure gradient, and the temperature, whereas the specific internal energy was expressed in terms of temperature, which is consistent with the Stefan problem formulation in [3, Chapter 5, § 9].

The Cauchy problem with periodic initial and boundary conditions for the above-mentioned Darcy–Stefan model is formulated and studied in the present paper; its physical essence is commented, and the boundaries of applicability of the model are determined. The Darcy–Stefan problem is studied in terms of an unknown specific internal energy rather than temperature, which makes the mathematical formulation much more difficult, because the energy balance equation with such a choice of the sought function is a degenerate parabolic-hyperbolic equation, the degeneration proceeding on a segment of values of the specific internal energy. A definition of the entropy solution of the Darcy–Stefan problem is introduced. This solution is more restrictive than the standard definition of the weak generalized solution. All possible entropy solutions are found to satisfy the second law of thermodynamics postulating non-negative production of entropy. From the physical viewpoint, this is an advantage of the entropy solution over the standard weak generalized solution. A theorem of the existence of the entropy solution is formulated; the proof of this theorem is based on the results of the Antontsev–Monakhov theory [4,

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Chapter 5] and on the method of the kinetic equation, which allows the initial entropy formulation to be interpreted in terms of the linear equation of the type of the Boltzmann equation in the kinetic theory of gases.

1. Formulation of the Darcy–Stefan Problem. Let us consider a thermomechanical system consisting of a motionless heat-conducting porous soil and a continuous medium completely filling the pores. The saturating continuous medium can be in a liquid or solid state and perform phase transitions between these states. The porous skeleton, i.e., soil, experiences no phase transitions.

A standard macroscopic approach (see, e.g., Eqs. (10.7.13)–(10.7.22) in [5]) is used to describe the system. The essence of this approach is as follows. At the level of a pore, the mathematical model, which unites the systems of the classical equations for the dynamics of the skeleton and saturating porous component and a system of relations on the pore surfaces, is replaced by an averaged system of equations, which describes the dynamics of a "homogenized" continuous medium whose thermomechanical properties differ from those of the solid skeleton and of the medium filling the pores. The regions occupied by the solid skeleton and the porous component are not distinguished, and the homogeneous model coefficients are quantities that bear information on both components of the thermomechanical system. These quantities are called the effective coefficients.

With allowance for some additional and simplifying assumptions, the most general model can be reduced to the following formulation of the Darcy–Stefan problem.

At each time instant $t \in [0,T]$ (T is an arbitrarily defined positive constant), the homogenized continuous medium is assumed to occupy a plane $\mathbb{R}^2$ or a three-dimensional space $\mathbb{R}^3$. Some part of the continuous medium with a temperature $\theta < 0$ occupies a certain domain $Y(t) := \{\boldsymbol{x} \in \mathbb{R}^d\colon\ \theta(\boldsymbol{x},t) < 0\}$ ($d = 2,3$); the remaining part of the continuous medium with a temperature $\theta > 0$ occupies the domain $W(t) := \mathbb{R}^d \setminus \overline{Y(t)} = \{\boldsymbol{x} \in \mathbb{R}^d\colon\ \theta(\boldsymbol{x},t) > 0\}$. The porous medium is in the solid (frozen) state in the domain $Y(t)$ and in the liquid (thawed) state in the domain $W(t)$. The locations of the domains $Y(t)$ and $W(t)$ and the interface $\Gamma(t) = \overline{Y(t)} \cap \overline{W(t)} = \{\boldsymbol{x} \in \mathbb{R}^d\colon\ \theta(\boldsymbol{x},t) = 0\}$ are unknown.

We have to find the distribution of the specific internal energy $e = e(\boldsymbol{x},t)$, the field of filtration velocities $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x},t)$, and the distribution of pressures $p_* = p_*(\boldsymbol{x},t)$, which satisfy the following equations and conditions:

— the energy balance equation

$$\frac{\partial e}{\partial t} + \operatorname{div}_x(\boldsymbol{v}e) = \Delta_x\theta, \qquad \boldsymbol{x} \in \mathbb{R}^d \setminus \Gamma(t), \quad t \in (0,T); \tag{1a}$$

— the thermodynamic equation of state of the continuous medium

$$\theta = \begin{cases} \theta_s(e), & e < 0, \\ 0, & 0 \le e \le l, \\ \theta_l(e), & e > l \end{cases} \tag{1b}$$

(the functions θ_s and θ_l are set so that $\theta = \theta(e)$ has a bounded second derivative and is a nonrigorously monotonically increasing function, with $0 < \theta_s'(e), \theta_l'(e) < +\infty\ \forall\, e \in \mathbb{R} \setminus [0,l]$);

— the condition of the absence of motion of the frozen phase

$$\boldsymbol{v} = 0, \qquad \boldsymbol{x} \in Y(t), \quad t \in (0,T); \tag{1c}$$

— the condition of continuity (condition of incompressibility) and the Darcy filtration law for $\theta \ge 0$

$$\operatorname{div}_x \boldsymbol{v} = 0, \qquad \boldsymbol{x} \in \overline{W(t)}, \quad t \in (0,T); \tag{1d}$$

$$\boldsymbol{v} = -\nabla_x p_* + \boldsymbol{g}(\theta), \qquad \boldsymbol{x} \in \overline{W(t)}, \quad t \in (0,T); \tag{1e}$$

— the equations of mass and heat balance on the interface $\Gamma(t)$ [6, § II.3]

$$\boldsymbol{v} \cdot \boldsymbol{n} = 0, \quad [\nabla_x\theta]_s^l \cdot \boldsymbol{n} = l\boldsymbol{V} \cdot \boldsymbol{n}, \qquad \boldsymbol{x} \in \Gamma(t), \quad t \in (0,T) \tag{1f}$$

(the second condition is also called the Stefan condition);

— the bounded periodic initial data for the distribution of the specific internal energy

$$e(\boldsymbol{x},0) = e_0(\boldsymbol{x}) \quad (|e_0(\boldsymbol{x})| \le c_0 = \text{const}), \qquad e_0(\boldsymbol{x} + \boldsymbol{k}_i) = e_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^d; \tag{1g}$$

— the conditions of spatial periodicity

$$e(\boldsymbol{x} + \boldsymbol{k}_i, t) = e(\boldsymbol{x},t), \quad \boldsymbol{v}(\boldsymbol{x} + \boldsymbol{k}_i, t) = \boldsymbol{v}(\boldsymbol{x},t), \qquad (\boldsymbol{x},t) \in \mathbb{R}^d \times [0,T]. \tag{1h}$$

In Eqs. (1a)–(1h), $\boldsymbol{k}_i$ $(1 \le i \le d)$ are the orthogonal unit vectors of the standard Cartesian basis in $\mathbb{R}^d$, θ is the temperature of the continuous medium, which is constant for specific internal energy values within the integer nondegenerate interval $[0, l]$, which means the phase transition in the porous component, $\theta = 0$ is the thawing (freezing) temperature, $e = l$ is the latent specific heat of melting, $\boldsymbol{g} \in C^2(\mathbb{R})$ is the buoyancy, which may be a nonlinear function of temperature in the general case [7], $\boldsymbol{n}$ is the unit vector normal to $\Gamma(t)$, which is directed toward $Y(t)$, $\boldsymbol{V}$ is the velocity of motion of $\Gamma(t)$, $[\nabla_x \theta]_s^l = (\nabla_x \theta)^l - (\nabla_x \theta)_s$ is the jump of the temperature gradient $\nabla_x \theta$ on the boundary $\Gamma(t)$, and $(\nabla_x \theta)^l$ and $(\nabla_x \theta)_s$ are the limiting values of $\nabla_x \theta$ on $\Gamma(t)$ in the domains $W(t)$ and $Y(t)$, respectively.

It should be noted that the Darcy–Stefan model (1a)–(1f) is a one-temperature model, i.e., the solid skeleton and the porous component have an identical temperature θ at all points of the spatial continuum, and spatial heat transfer is described by one energy balance equation (1a). A more general model of liquid filtration through porous soils (see, e.g., Eqs. (10.7.13)–(10.7.22) in [5]) is a two-temperature model and contains two energy balance equations: for the soil skeleton and for the saturating liquid. In some cases, first of all, at low Reynolds numbers (i.e., for rather slow filtration flows), the difference between the temperatures of the soil skeleton and the porous medium can be neglected [5, pp. 646–647], because the time needed for rough equalization of the temperatures in the skeleton and porous medium in small spatial volumes is insignificant, as compared with the duration of heat income due to filtration flows. In particular, an example of such a case is the process of natural freezing (thawing) of soils. Hence, the energy balance equations for two temperatures are reduced to one energy balance equation for one temperature, i.e., to Eq. (1a). Note that the coefficients in Eq. (1a) are averaged (effective), i.e., they depend on specific heats, thermal conductivities, densities, and specific volumes of both the porous component and the solid skeleton. (To simplify the calculations, we assume that the coefficients in convective terms are equal to unity.)

2. Concept of the Entropy Solution. Theorem of Existence of Entropy Solutions. To formulate the concept of the entropy solution of the Darcy–Stefan problem, we apply the following notation for the linear spaces of periodic functions: $Q = \Omega \times (0, T)$; spatial period $\Omega := [0, 1)^d$; Banach spaces $L^p \subset L^p_{\text{loc}}(\mathbb{R}^d)$ and $H^{s,p} \subset H^{s,p}_{\text{loc}}(\mathbb{R}^d)$ consisting of 1-periodic functions and supplemented with the norms $\|u\|_{L^p} = \|u\|_{L^p(\Omega)}$ and $\|u\|_{H^{s,p}} = \|u\|_{H^{s,p}(\Omega)}$; for integer $m \ge 0$, closed subspace C^m of 1-periodic functions with respect to $\boldsymbol{x}$ from $C^m(\mathbb{R}^d)$.

Let us introduce the concept of the entropy solution of the Darcy–Stefan problem.

Definition 1. The entropy solution of the Darcy–Stefan problem is understood as a pair of functions (e, p_*) if these functions satisfy the following conditions and relations:

1) condition of regularity

$$e \in L^\infty(Q), \quad \theta(e), H(e) \in L^2(0, T; H^{1,2}), \quad p_* \in L^2(0, T; H^{2,2}), \tag{2a}$$

where $H(e) \overset{\text{def}}{=} \int^e \sqrt{\theta'(\lambda)}\, d\lambda$;

2) integral inequality

$$\int_Q \Big\{ \varphi(e)\, \partial_t \zeta + \varphi^+(e)[-\nabla_x p_* + \boldsymbol{g}(\theta(e))] \cdot \nabla_x \zeta + w(e)\, \Delta_x \zeta - \varphi''(e) |\nabla_x H(e)|^2 \zeta \Big\}\, d\boldsymbol{x}\, dt$$

$$+ \int_\Omega \varphi(e_0) \zeta(\boldsymbol{x}, 0)\, d\boldsymbol{x} \ge 0, \tag{2b}$$

where φ, φ^+, and w are arbitrary functions, such that

$$\varphi \in C^2_{\text{loc}}(\mathbb{R}), \quad \varphi''(e) \ge 0, \quad \varphi^+(e) = \int^e I_{\lambda \ge 0} \varphi'(\lambda)\, d\lambda, \quad w(e) = \int^e \varphi'(\lambda) \theta'(\lambda)\, d\lambda, \tag{2c}$$

$\zeta \in C^2_{\text{loc}}(\mathbb{R}^d \times [0, T])$ is an arbitrary non-negative function 1-periodic with respect of $\boldsymbol{x}$, which vanishes in the neighborhood $\{t = T\}$;

3) equation

$$\Delta_x p_* = \text{div}_x \{\boldsymbol{g}(\theta(e))\} \quad \text{almost everywhere in } Q. \tag{2d}$$

Note, if the entropy solution is constructed, then the velocity vector can be found by the formula

$$\boldsymbol{v}(\boldsymbol{x},t) = I_{e(x,t)\geq 0}\Big[-\nabla_x p_*(\boldsymbol{x},t) + \boldsymbol{g}(\theta(e(\boldsymbol{x},t)))\Big]. \tag{3}$$

Though the distribution of pressure p_* in determining the entropy solution is found within the entire period Ω, it is physically reasonable to consider only the values for $\boldsymbol{x} \in \Omega \setminus Y(t)$. With allowance for the equation of state (1b), formula (3) is in good agreement with (1c) and (1e).

In Eqs. (2c) and (3), and further on, $I_{\lambda\geq s}$ denotes the Heaviside function of the variable λ with a jump at the point $\lambda = s$:

$$I_{\lambda\geq s} \stackrel{\text{def}}{=} \begin{cases} 1, & \lambda \geq s, \\ 0, & \lambda < s. \end{cases}$$

With allowance for Eqs. (1d) and (3), and the initial condition (1g), the integral inequality (2b) in the sense of the theory of distributions is equivalent to the inequality

$$\frac{\partial\varphi(e)}{\partial t} + \boldsymbol{v}\cdot\nabla_x\varphi(e) - \Delta_x w(e) \leq -\varphi''(e)|\nabla_x H(e)|^2 \quad \text{in } D'(Q). \tag{4}$$

Under the assumption that $\varphi(e) = \pm e$, the energy balance equation (1a) is valid in the entire space Q in the sense of distributions. As was stated in [6, § II.3], Eq. (1a) defined in the entire space Q is formally satisfied in each of the regions $\{(\boldsymbol{x},t) \in Q\colon\ \boldsymbol{x} \in Y(t),\ t \geq 0\}$, and $\{(\boldsymbol{x},t) \in Q\colon\ \boldsymbol{x} \in W(t),\ t \geq 0\}$, and the Stefan condition [i.e., the second equation in (1f)] is satisfied on the interface surface $\Gamma(t)$. For this reason, and also by virtue of the regularity conditions (2a), all possible entropy solutions of the Darcy–Stefan problem determined by conditions (2a)–(2d) are weak solutions of the Darcy–Stefan problem, i.e., they satisfy the equations of system (1a)–(1h), but Eq. (1a) is satisfied in the weak sense, and conditions (1f) are satisfied in the sense of the traces. The choice $\varphi(e) = \pm e$, however, is a particular case; hence, the integral inequality (2b) [and, correspondingly, the differential inequality (4)] is more restrictive than the energy balance equation. In essence, in the case of smooth and convex functions $\varphi(e)$ that are not equal to e and $-e$, inequality (4) is a complement to the formulation of the Darcy–Stefan problem.

Let us comment on the physical motivation of this complement. Integrating inequality (4) with respect to $\boldsymbol{x}$ and t on $\Omega\times(0,\tau)$ and taking into account the solenoidal character of $\boldsymbol{v}$, 1-periodicity of $\boldsymbol{v}$ and e with respect to $\boldsymbol{x}$, and Green's formulas for an arbitrary $\tau > 0$, we obtain

$$\int_\Omega \varphi(e(\boldsymbol{x},\tau))\,d\boldsymbol{x} + \int_0^\tau\int_\Omega \varphi''(e)|\nabla_x H(e)|^2\,d\boldsymbol{x}\,dt \leq \int_\Omega \varphi(e_0(\boldsymbol{x}))\,d\boldsymbol{x} \quad \forall\,\varphi \in C^2(\mathbb{R}),\ \varphi'' \geq 0. \tag{5}$$

Requiring the axiom of the state of liquids and gases [8, Part II, § 8] to be satisfied for the problem considered, i.e., postulating the fundamental thermodynamic identity $\theta_{abs}\,dS = de$ [an incompressible and one-parameter medium is considered; $p_* d(1/\rho) \equiv 0$, where ρ is a constant density], we can present the specific entropy S as a function of the specific internal energy in the form

$$S(e) = \int^e \frac{d\lambda}{\theta(\lambda) + \theta_{dif}}. \tag{6}$$

In Eq. (6) and in the fundamental thermodynamic identity, θ_{abs} is the absolute temperature and θ_{dif} is the difference between the zero value of temperature on the scale θ and the absolute zero, i.e., $\theta_{abs}(e) = \theta(e) + \theta_{dif}$. Twice differentiating Eq. (6) with respect to e, by virtue of the equation of state (1b), we find that $S''(e) \leq 0$ for all $e \in \mathbb{R}$. Hence, we can set $\varphi(e) = -S(e)$ in inequalities (2b), (4), and (5). Then it follows from Eq. (5) that

$$\int_\Omega S(e(\boldsymbol{x},\tau))\,d\boldsymbol{x} \geq \int_\Omega S(e_0(\boldsymbol{x}))\,d\boldsymbol{x}$$

for all $\tau > 0$. This inequality coincides with the second law of thermodynamics: entropy production is non-negative.

The above-performed considerations also imply that an arbitrary function $S(e) = -\varphi(e)$, where φ is a smooth and convex function, can be used as the entropy in the examined thermomechanical system if we do not require the axiom of the state of liquids and gases to be satisfied. Then the satisfaction of the second law of thermodynamics is guaranteed by inequality (4) [inequality (2b) in the definition of the entropy solution], which is further called the entropy inequality.

It should be noted that the entropy is understood in the mathematical theory of nonlinear conservation laws (see, e.g., [9, 10]) as a convex function φ rather than the function with the opposite sign $S = -\varphi$, as it is commonly used in thermodynamics.

The following theorem is stated.

Theorem 1. *For all initial data $e_0 \in L^\infty$, such that $-c_0 \le e_0(\boldsymbol{x}) \le c_0$ almost everywhere in $\mathbb{R}^d$, the Darcy–Stefan problem has at least one entropy solution.*

Theorem 1 is proved in Secs. 3–5.

3. Parabolic Approximation of the Darcy–Stefan Problem. Partial Compactness of Approximate Solutions. Along with the Darcy–Stefan problem, we consider its parabolic approximation

$$\frac{\partial e}{\partial t} + \boldsymbol{u}\cdot\nabla_x e^+ = \Delta_x\theta(e) + \varepsilon\Delta_x e, \qquad (\boldsymbol{x},t)\in\mathbb{R}^d\times(0,T); \tag{7a}$$

$$\boldsymbol{u} = -\nabla_x p_* + \boldsymbol{g}(\theta(e)), \quad \operatorname{div}_x \boldsymbol{u} = 0, \qquad (\boldsymbol{x},t)\in\mathbb{R}^d\times(0,T); \tag{7b}$$

$$e(\boldsymbol{x}+\boldsymbol{k}_i,t) = e(\boldsymbol{x},t), \quad \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{k}_i,t) = \boldsymbol{u}(\boldsymbol{x},t), \qquad (\boldsymbol{x},t)\in\mathbb{R}^d\times(0,T) \tag{7c}$$

with the initial data (1g) [in (7a), $e^+ = I_{e\ge 0}\, e$].

According to the theory of filtration of immiscible liquids [4, Chapter 5], for an arbitrary fixed $\varepsilon > 0$, there exists a unique smooth solution $(e_\varepsilon, p_{*\varepsilon}, \boldsymbol{u}_\varepsilon)$ of problem (7a)–(7c), (1g); the pressure $p_{*\varepsilon}$ is determined with accuracy to a constant term, which can be fixed by a standard requirement, e.g., $\int\limits_\Omega p_{*\varepsilon}(\boldsymbol{x},t)\,d\boldsymbol{x} = 0$. Based on the maximum principle and energy estimates, we obtain

$$-c_0 \le e_\varepsilon(\boldsymbol{x},t) \le c_0 \qquad \text{in} \quad \mathbb{R}^d\times(0,T); \tag{8a}$$

$$\|\nabla_x\theta(e_\varepsilon)\|^2_{L^2(Q)} + \|\nabla_x H(e_\varepsilon)\|^2_{L^2(Q)} + \varepsilon\|\nabla_x e_\varepsilon\|^2_{L^2(Q)} + \|\nabla_x p_{*\varepsilon}\|^2_{L^2(Q)} \le c_1(Q), \tag{8b}$$

where the constant c_1 is independent of ε.

Inequalities (8) imply the existence of a sequence of solutions e_ε, $p_{*\varepsilon}$, $\boldsymbol{u}_\varepsilon$ of problem (7a)–(7c), (1g) and five functions e, p_*, $\boldsymbol{u}$, θ_*, and H_* such that the following limit relations are valid for $\varepsilon \searrow 0$:

$$e_\varepsilon \to e \qquad \text{weak star in} \quad L^\infty(Q); \tag{9}$$

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u}, \quad \nabla_x p_{*\varepsilon} \to \nabla_x p_* \qquad \text{weakly in} \quad L^2(Q); \tag{10}$$

$$\nabla_x\theta(e_\varepsilon) \to \nabla_x\theta_*, \quad \nabla_x H(e_\varepsilon) \to \nabla_x H_* \qquad \text{weakly in} \quad L^2(Q). \tag{11}$$

As the Darcy–Stefan problem is nonlinear, to pass to the limit in approximate equations, we have to prove strong convergence of some subsequence of approximate solutions. For this purpose, we prove the pre-compactness of the families $\{\nabla_x p_{*\varepsilon}\}_{\varepsilon>0}$ and $\{\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$.

Statement 1. *For an arbitrary bounded set $K \subset \mathbb{R}^d_x$ with a sufficiently smooth boundary, there exists a constant $c_2(K)$ such that*

$$\|\nabla_x p_{*\varepsilon}\|_{L^2(0,T;H^{1,2}(K))} + \|\partial_t\nabla_x p_{*\varepsilon}\|_{L^2(0,T;H^{-1,2}(K))}$$

$$+ \|\boldsymbol{u}_\varepsilon\|_{L^2(0,T;H^{1,2}(K))} + \|\partial_t\boldsymbol{u}_\varepsilon\|_{L^2(0,T;H^{-1,2}(K))} \le c_2(K). \tag{12}$$

In $L^2_{\mathrm{loc}}(\mathbb{R}^d_x\times(0,T))$, the families $\{\nabla_x p_{*\varepsilon}\}_{\varepsilon>0}$ and $\{\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ are relatively compact.

The proof is based on the use of a standard technique for constructing *a priori* estimates of solutions of parabolic and elliptic equations [3, 11].

Let K be an arbitrary bounded subset $\mathbb{R}^d_x$ with a sufficiently smooth boundary. Equations (7b) yield Eq. (2d), which, in turn, by virtue of inequalities (8) and uniform boundedness of $\boldsymbol{g}'_\theta(\theta(e_\varepsilon))$, implies that

$$\|\Delta_x p_{*\varepsilon}\|_{L^2(K\times(0,T))} \le c_3(K) \tag{13}$$

[hereinafter in the proof of this statement, the constants $c_j(K)$ $(j = 3, 4, \dots, 9)$ are independent of ε]. Estimates (8) and (13) and the second fundamental inequality for elliptic operators [11, Chapter II, § 6] yield the estimate

$$\|p_{*\varepsilon}\|_{L^2(0,T;H^{2,2}(K))} \le c_4(K). \tag{14}$$

Multiplying (7a) by the function $g'_{i\theta}(\theta(e_\varepsilon))\theta'(e_\varepsilon)$ $(1 \le i \le d)$ and conducting some simple transformations, we obtain

$$\partial_t g_i(\theta(e_\varepsilon)) = -\operatorname{div}_x(\boldsymbol{u}h_i(e_\varepsilon)) + \Delta_x r_i(e_\varepsilon) + \varepsilon\Delta_x g_i(\theta(e_\varepsilon)) - \theta'(e_\varepsilon)g''_{i\theta\theta}(\theta(e_\varepsilon))|\nabla_x\theta(e_\varepsilon)|^2$$
$$- g'_{i\theta}(\theta(e_\varepsilon))\theta''(e_\varepsilon)|\nabla_x H(e_\varepsilon)|^2 - \varepsilon g''_{i\theta\theta}(\theta(e_\varepsilon))|\nabla_x\theta(e_\varepsilon)|^2 - \varepsilon g'_{i\theta}(\theta(e_\varepsilon))\theta''(e_\varepsilon)|\nabla_x e_\varepsilon|^2, \tag{15}$$

where $h'_i(e_\varepsilon) = I_{e_\varepsilon \ge l}\, g'_{i\theta}(\theta(e_\varepsilon))\theta'(e_\varepsilon)$ and $r'_i(e_\varepsilon) = g'_{i\theta}(\theta(e_\varepsilon))(\theta'(e_\varepsilon))^2$. Multiplying both sides of equality (15) by an arbitrary function $\varphi \in L^2(0,T;C^1_0(K))$, integrating the resultant equality with respect to $K\times(0,T)$, and integrating the right side by parts, we find

$$\int\limits_{K\times(0,T)} \Big(\partial_t g_i(\theta(e_\varepsilon))\Big)\varphi\, d\boldsymbol{x}\, dt = \int\limits_{K\times(0,T)} \boldsymbol{u}h_i(e_\varepsilon)\cdot\nabla_x\varphi\, d\boldsymbol{x}\, dt$$

$$- \int\limits_{K\times(0,T)} g'_{i\theta}(\theta(e_\varepsilon))\theta'(e_\varepsilon)\nabla_x\theta(e_\varepsilon)\cdot\nabla_x\varphi\, d\boldsymbol{x}\, dt - \varepsilon \int\limits_{K\times(0,T)} g'_{i\theta}(\theta(e_\varepsilon))\theta'(e_\varepsilon)\nabla_x e_\varepsilon\cdot\nabla_x\varphi\, d\boldsymbol{x}\, dt$$

$$- \int\limits_{K\times(0,T)} g''_{i\theta\theta}(\theta(e_\varepsilon))\theta'(e_\varepsilon)|\nabla_x\theta(e_\varepsilon)|^2\, d\boldsymbol{x}\, dt - \int\limits_{K\times(0,T)} g'_{i\theta}(\theta(e_\varepsilon))\theta''(e_\varepsilon)|\nabla_x H(e_\varepsilon)|^2\, d\boldsymbol{x}\, dt$$

$$- \varepsilon \int\limits_{K\times(0,T)} g''_{i\theta\theta}(\theta(e_\varepsilon))|\nabla_x\theta(e_\varepsilon)|^2\, d\boldsymbol{x}\, dt - \varepsilon \int\limits_{K\times(0,T)} g'_{i\theta}(\theta(e_\varepsilon))\theta''(e_\varepsilon)|\nabla_x e_\varepsilon|^2\, d\boldsymbol{x}\, dt.$$

Applying the Cauchy inequality in the right side and using estimates (8), we obtain

$$\Big| \int\limits_{K\times(0,T)} \Big(\partial_t g_i(\theta(e_\varepsilon))\Big)\varphi\, d\boldsymbol{x}\, dt\Big| \le c_5(K)\|\varphi\|_{L^2(0,T;H^{1,2}(K))},$$

whence it follows

$$\|\partial_t g_i(\theta(e_\varepsilon))\|_{L^2(0,T;H^{-1,2}(K))} \le c_6(K) \qquad (1 \le i \le d). \tag{16}$$

By virtue of the properties of elliptic operators [11, Chapter II], inequality (16) and the equality $\partial_t\Delta_x p_{*\varepsilon} = \partial_t \operatorname{div}_x \boldsymbol{g}(\theta(e_\varepsilon))$ yield the estimate

$$\|\partial_t\nabla_x p_{*\varepsilon}\|_{L^2(0,T;H^{-1,2}(K))} \le c_7(K). \tag{17}$$

From Eq. (7b), estimate (14), inclusion $\nabla_x\boldsymbol{g}(\theta(e_\varepsilon)) \in L^2(K\times(0,T))$, equation $\partial_t\boldsymbol{u}_\varepsilon = -\partial_t\nabla_x p_{*\varepsilon} + \partial_t\boldsymbol{g}(\theta(e_\varepsilon))$, and estimates (16) and (17), there follow the inequalities

$$\|\boldsymbol{u}_\varepsilon\|_{L^2(0,T;H^{1,2}(K))} \le c_8(K), \qquad \|\partial_t\boldsymbol{u}_\varepsilon\|_{L^2(0,T;H^{-1,2}(K))} \le c_9(K). \tag{18}$$

Thus, estimate (12) is a consequence of estimates (14), (17), and (18). It should be noted that the families $\{\nabla_x p_{*\varepsilon}\}_{\varepsilon>0}$ and $\{\boldsymbol{u}_\varepsilon\}_{\varepsilon>0}$ are relatively compact owing to (12) and the results on compactness from [12].

4. Kinetic Formulation Associated with the Darcy–Stefan Problem. 4.1. *Preliminary Information.* In the present work, the compactness of subsequences of approximate solutions is justified and the solvability of the Darcy–Stefan problem is proved on the basis of constructing a kinetic equation associated with the problem, which is a tool of the method of the kinetic equation. This method was developed to study a wide range of applied problems, for instance, boundary-value problems for the system of equations of isentropic gas dynamics, quasi-linear conservation laws of the first and second order, and models of two-phase filtration in fibrous structures [9, 13, 14]. The method of the kinetic equation allows quasi-linear equations to be reduced to linear scalar equations whose solutions are functions of "distributions" containing additional "kinetic" variables.

As the kinetic formulation associated with the Darcy–Stefan problem is constructed similar to [13, 14] and includes the notion of measure-valued mappings corresponding to weakly converging sequences of approximate solutions, we introduce some definitions and notations for measure-valued mappings: $\mathrm{Prob}\,(\mathbb{R}^n)$ is the set of probability measures in $\mathbb{R}^n$, i.e., non-negative Radon measures with a unit norm; $\mathbb{M}(\mathbb{R}^n)$ is the space of finite Radon measures above $\mathbb{R}^n$. The norm in $\mathbb{M}(\mathbb{R}^n)$ is introduced in a standard manner [10, Sec. 1.2.8]. The mapping μ: $\mathbb{R}^d_x \times (0,T) \mapsto \mathbb{M}(\mathbb{R}^n)$ is called bounded, weak star measurable, and 1-periodic with respect to $\boldsymbol{x}$, if, for an arbitrary $F \in L^1_{\mathrm{loc}}(\mathbb{R}^d_x \times (0,T); C_0(\mathbb{R}^n))$, the function $(\boldsymbol{x},t) \mapsto \int\limits_{\mathbb{R}^n_p} F(\boldsymbol{x},t,\boldsymbol{p})\,d\mu_{x,t}(\boldsymbol{p})$ is measurable, and the following equality is valid for $1 \le i \le d$:

$$\int\limits_{\mathbb{R}^n_p} F(\boldsymbol{x},t,\boldsymbol{p})\,d\mu_{x+k_i,t}(\boldsymbol{p}) = \int\limits_{\mathbb{R}^n_p} F(\boldsymbol{x}-\boldsymbol{k}_i,t,\boldsymbol{p})\,d\mu_{x,t}(\boldsymbol{p}).$$

Here $\mu_{x,t} = \mu(\boldsymbol{x},t)$ is a standard notation, i.e., the measures $\mu_{x,t}$ are parameterized with parameters $\boldsymbol{x}$ and t. Following definition 2.7 in [10, Chapter 3], we denote the linear space consisting of the above-described measure-valued mappings μ by $L^\infty_w(\mathbb{R}^d_x \times (0,T); \mathbb{M}(\mathbb{R}^n))$ and introduce the norm in this space

$$\|\mu\|_{L^\infty_w(\mathbb{R}^d_x \times (0,T);\mathbb{M}(\mathbb{R}^n))} = \operatorname*{ess\,sup}_{(x,t)\in\mathbb{R}^d_x\times(0,T)} \|\mu_{x,t}\|_{\mathbb{M}(\mathbb{R}^n)}.$$

4.2. *Concept of the Kinetic Formulation.* The limit relations (9) and (11), the Tartar theorem (Theorem 2.3 in [10, Chapter 3]) and the Ball theorem (Theorem 2.1 in [10, Chapter 4]) imply the existence of a subsequence $\{e_\varepsilon\}_{\varepsilon\to 0}$ and 1-periodic (with respect to $\boldsymbol{x}$) measure-valued functions $\nu \in L^\infty_w(\mathbb{R}^d_x \times (0,T); \mathrm{Prob}\,(\mathbb{R}_\lambda))$ and $\sigma \in L^\infty_w(\mathbb{R}^d_x \times (0,T); \mathrm{Prob}\,(\mathbb{R}_\lambda \times \mathbb{R}^d_q))$, such that

$$\varphi(e_\varepsilon) \underset{\varepsilon\searrow 0}{\longrightarrow} \int\limits_{\mathbb{R}_\lambda} \varphi(\lambda)\,d\nu_{x,t}(\lambda) \qquad \text{weak star in} \quad L^\infty(\mathbb{R}^d_x \times (0,T)) \tag{19}$$

for all functions $\varphi \in C(\mathbb{R}_\lambda)$ and

$$\psi(e_\varepsilon, \nabla_x H(e_\varepsilon)) \underset{\varepsilon\searrow 0}{\longrightarrow} \int\limits_{\mathbb{R}_\lambda\times\mathbb{R}^d_q} \psi(\lambda,\boldsymbol{q})\,d\sigma_{x,t}(\lambda,\boldsymbol{q}) \qquad \text{weak star in} \quad L^\infty(\mathbb{R}^d_x \times (0,T)) \tag{20}$$

for all functions $\psi \in C(\mathbb{R}_\lambda \times \mathbb{R}^d_q)$ satisfying the condition $|\psi(\lambda,\boldsymbol{q})| \le c(1+|\lambda|+|\boldsymbol{q}|)^r$, $0 \le r < 2$. The measures $\nu_{x,t}$ and $\sigma_{x,t}$ are called the Young measures associated with weakly converging subsequences $\{e_\varepsilon\}$ and $\{e_\varepsilon, \nabla_x H(e_\varepsilon)\}$, respectively.

We introduce the distribution function of the measure $\nu_{x,t}$

$$f(\boldsymbol{x},t,\lambda) = \int\limits_{\mathbb{R}_s} I_{\lambda\ge s}\,d\nu_{x,t}(s) \tag{21}$$

and the parameterized Heaviside function

$$f_\varepsilon(\boldsymbol{x},t,\lambda) = I_{\lambda \ge e_\varepsilon(x,t)}, \tag{22}$$

which is a distribution function of the parameterized measure $\gamma_{e_\varepsilon(x,t)}$ — the Dirac measure on $\mathbb{R}_\lambda$ concentrated at the point $\lambda = e_\varepsilon(\boldsymbol{x},t)$. By virtue of Eq. (19) and the obvious presentation

$$\varphi(e_\varepsilon(\boldsymbol{x},t)) = -\int\limits_{\mathbb{R}} \varphi'(\lambda) f_\varepsilon(\boldsymbol{x},t,\lambda)\,d\lambda \qquad \forall\, \varphi \in C^1_0(\mathbb{R}),$$

we have the limit relation

$$f_\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} f \qquad \text{weak star in} \quad L^\infty((0,T)\times\mathbb{R}_\lambda; L^\infty). \tag{23}$$

By virtue of Statement 1, there exists another subsequence from $\{e_\varepsilon, \boldsymbol{u}_\varepsilon, p_{*\varepsilon}\}$ such that

$$\boldsymbol{u}_\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} \boldsymbol{u}, \quad \nabla_x p_{*\varepsilon} \underset{\varepsilon\searrow 0}{\longrightarrow} \nabla_x p_* \qquad \text{strongly in} \quad L^2(Q). \tag{24}$$

Then, we denote the parameterized Radon measure on $\mathbb{R}_\lambda$ and the Radon measure on $Q \times \mathbb{R}_\lambda$ described by the formulas

$$d_\lambda \chi_\varepsilon(\boldsymbol{x}, t, \lambda) = |\nabla_x H(e_\varepsilon)|^2 d\gamma_{e_\varepsilon(x,t)}(\lambda) \qquad \text{almost everywhere} \quad (\boldsymbol{x}, t) \in \mathbb{R}^d_x \times (0, T); \tag{25}$$

$$dM_\varepsilon(\boldsymbol{x}, t, \lambda) = \varepsilon |\nabla_x e_\varepsilon|^2 \, d\gamma_{e_\varepsilon(x,t)}(\lambda) \, d\boldsymbol{x} \, dt, \tag{26}$$

by $d_\lambda \chi_\varepsilon(\,\cdot\,, \cdot\,, \lambda)$ and M_ε, respectively. By virtue of estimates (8), definition of the Young measure $\sigma_{x,t}$, and Lemmas 9 and 10 in [13], the following statement is valid.

Statement 2. *There exists a subsequence from* $\{e_\varepsilon\}$ *such that:*

1) *the function*

$$\chi(\boldsymbol{x}, t, \lambda) := \int\limits_{(-\infty, \lambda] \times \mathbb{R}^d_q} |\boldsymbol{q}|^2 \, d\sigma_{x,t}(s, \boldsymbol{q}) \tag{27}$$

is determined almost everywhere in $(\boldsymbol{x}, t) \in \mathbb{R}^d \times (0, T)$*, 1-periodic with respect to* $\boldsymbol{x}$*, continuous on the right with respect to* λ*, and has a finite limit as* $\lambda \to \infty$ *almost everywhere in* $(\boldsymbol{x}, t) \in \mathbb{R}^d \times (0, T)$*;*

2) *the support of the Stieltjes measure* $d_\lambda \chi(\,\cdot\,, \cdot\,, \lambda)$ *belongs to the segment* $[-c_0, c_0]$ *almost everywhere in* $(\boldsymbol{x}, t) \in \mathbb{R}^d \times (0, T)$*, and the weak star measurable mapping* $(\boldsymbol{x}, t) \mapsto d_\lambda \chi(\boldsymbol{x}, t, \lambda)$ *belongs to the space* $L^1_w(Q; \mathbb{M}(\mathbb{R}_\lambda))$ *and is related to the sequence of weak star measurable mappings* $\{(\boldsymbol{x}, t) \mapsto d_\lambda \chi_\varepsilon(\boldsymbol{x}, t, \lambda)\}_{\varepsilon \to 0}$ *by the limit relation*

$$d_\lambda \chi_\varepsilon \underset{\varepsilon \searrow 0}{\longrightarrow} d_\lambda \chi \qquad \textit{weak star in} \quad L^1_w(Q; \mathbb{M}(\mathbb{R}_\lambda)); \tag{28}$$

3) *there exists a non-negative, 1-periodic with respect to* $\boldsymbol{x}$ *measure* $M \in \mathbb{M}(Q \times \mathbb{R}_\lambda)$ *related to the sequence of measures* $\{M_\varepsilon\}_{\varepsilon \to 0}$ *by the limit relation*

$$M_\varepsilon \underset{\varepsilon \searrow 0}{\longrightarrow} M \qquad \textit{weak star in} \quad \mathbb{M}(Q \times \mathbb{R}_\lambda). \tag{29}$$

In formulation 2 of Statement 2, $L^1_w(Q; \mathbb{M}(\mathbb{R}_\lambda))$ denotes the space of weak star measurable mappings μ: $Q \mapsto \mathbb{M}(\mathbb{R}_\lambda)$ such that the integral $\displaystyle\int\limits_Q \Big| \int\limits_{\mathbb{R}_\lambda} F(\boldsymbol{x}, t, \lambda) \, d\nu_{x,t}(\lambda) \Big| \, d\boldsymbol{x} \, dt$ is finite for all $F \in L^\infty(Q; C_0(\mathbb{R}_\lambda))$. Correspondingly, the limit relation (28) is understood in the sense of convergence of the integrals

$$\int\limits_Q \int\limits_{\mathbb{R}_\lambda} F(\boldsymbol{x}, t, \lambda) \, d_\lambda \chi_\varepsilon(\boldsymbol{x}, t, \lambda) \, d\boldsymbol{x} \, dt \underset{\varepsilon \searrow 0}{\longrightarrow} \int\limits_Q \int\limits_{\mathbb{R}_\lambda} F(\boldsymbol{x}, t, \lambda) \, d_\lambda \chi(\boldsymbol{x}, t, \lambda) \, d\boldsymbol{x} \, dt \quad \forall\, F \in L^\infty(Q; C_0(\mathbb{R}_\lambda)).$$

Using the limit relations (23), (24), (28), and (29), we can pass to the limit as $\varepsilon \searrow 0$ (by choosing an appropriate subsequence if necessary) in the equations of the approximating problem (7a)–(7c), (1g). Following Theorem 5 in [13], we derive the limit formulation (called the kinetic formulation of the Darcy–Stefan problem) and, simultaneously, a theorem of solvability of this formulation.

Theorem 2. *Let* $(e_\varepsilon, \boldsymbol{u}_\varepsilon, p_{*\varepsilon})$ *be a solution of problem* (7a)–(7c)*,* (1g)*, and the distribution function* f_ε *and the measures* $d_\lambda \chi_\varepsilon$ *and* M_ε *be defined by Eqs.* (22)*,* (25)*, and* (26)*. There exists a sequence of small parameters* $\varepsilon = \varepsilon_k \underset{k \to \infty}{\longrightarrow} 0$ *such that the sequence* $\{f_{\varepsilon_k}, \boldsymbol{u}_{\varepsilon_k}, p_{*\varepsilon_k}, d_\lambda \chi_{\varepsilon_k}, M_{\varepsilon_k}\}_{k=1,2,\ldots}$ *for* $k \to \infty$ *converges to five functions and measures* f*,* $\boldsymbol{u}$*,* p_**,* $d_\lambda \chi$*, and* M*, these functions and measures being a solution of problem K considered below* [*the convergence* $(f_{\varepsilon_k}, \boldsymbol{u}_{\varepsilon_k}, p_{*\varepsilon_k}, d_\lambda \chi_{\varepsilon_k}, M_{\varepsilon_k}) \underset{k \to \infty}{\longrightarrow} (f, \boldsymbol{u}, p_*, d_\lambda \chi, M)$ *is understood in the sense of the limiting relations* (23)*,* (24)*,* (28)*, and* (29)].

Problem K (kinetic formulation of the Darcy–Stefan problem)**.** With a given initial distribution function $f_0(\boldsymbol{x}, \lambda) = I_{\lambda \geq e_0(x)}$, where e_0 is defined by condition (1g), we have to find a distribution function $f \in L^\infty(\mathbb{R}^d \times (0, T) \times \mathbb{R}_\lambda)$, a vector field $\boldsymbol{u} \in L^2(0, T; H^{1,2})$, a pressure function $p_* \in L^2(0, T; H^{2,2})$, a kinetic measure of parabolic dissipation $d_\lambda \chi \in L^1_w(Q; \mathbb{M}(\mathbb{R}_\lambda))$, and a kinetic entropy measure of defect $M \in \mathbb{M}(Q \times \mathbb{R}_\lambda)$ that satisfy the following requirements.

1. The function $f(\boldsymbol{x}, t, \lambda)$ is 1-periodic with respect to $\boldsymbol{x}$, monotonic, and continuous on the right with respect to $\lambda \in \mathbb{R}$. Thereby,

$$f(\boldsymbol{x}, t, \lambda) = 0 \quad \text{for} \quad \lambda < -c_0, \qquad f(\boldsymbol{x}, t, \lambda) = 1 \quad \text{for} \quad \lambda \geq c_0.$$

In particular, $0 \leq f \leq 1$ almost everywhere in $Q \times \mathbb{R}_\lambda$. Hence, the Stieltjes measure $\nu_{x,t} = d_\lambda f(\boldsymbol{x}, t, \lambda)$ is a probability measure on $\mathbb{R}_\lambda$ and $\operatorname{spt} \nu_{x,t} \subset [-c_0, c_0]$.

2. The weak star measurable mapping $(\boldsymbol{x}, t) \mapsto d_\lambda \chi(\boldsymbol{x}, t, \lambda)$ is non-negative and 1-periodic with respect to $\boldsymbol{x}$, and the support of the measure $d_\lambda \chi(\,\cdot\,, \cdot\,, \lambda)$ belongs to the segment $[-c_0, c_0] \subset \mathbb{R}_\lambda$ almost everywhere in $(\boldsymbol{x}, t) \in \mathbb{R}^d_x \times (0, T)$.

3. There exists a weak star measurable mapping $\sigma \in L^\infty_w(\mathbb{R}^d_x \times (0, T); \operatorname{Prob}(\mathbb{R}_\lambda \times \mathbb{R}^d_q))$ such that:
— its projection onto the space of the variable λ coincides with $d_\lambda f$

$$\int\limits_{\mathbb{R}_\lambda} \zeta(\lambda)\, d_\lambda f(\boldsymbol{x}, t, \lambda) = \int\limits_{\mathbb{R}_\lambda \times \mathbb{R}^d_q} \zeta(\lambda)\, d\sigma_{x,t}(\lambda, \boldsymbol{q}) \quad \forall\, \zeta \in C(\mathbb{R}_\lambda) \quad \text{almost everywhere} \quad (\boldsymbol{x}, t) \in \mathbb{R}^d_x \times (0, T);$$

— by means of integral (27), the mapping generates the kinetic measure of parabolic dissipation $d_\lambda \chi$;
— the mapping is related to f by an additional relation

$$\sqrt{\theta'(\lambda)}\, \nabla_x f(\boldsymbol{x}, t, \lambda) = -\int\limits_{\mathbb{R}^d_q} \boldsymbol{q}\, d\sigma_{x,t}(\lambda, \boldsymbol{q})$$

understood in the sense of distributions.

4. The kinetic entropy measure of defect M is non-negative and 1-periodic with respect to $\boldsymbol{x}$.

5. The functions and measures f, $\boldsymbol{u}$, p_*, $d_\lambda \chi$, and M satisfy the equations

$$\frac{\partial f}{\partial t} + I_{\lambda \geq 0}\, \boldsymbol{u} \cdot \nabla_x f - \theta'(\lambda)\, \Delta_x f + \frac{\partial}{\partial \lambda}\, (d_\lambda \chi + M) = 0, \quad (\boldsymbol{x}, t, \lambda) \in Q \times \mathbb{R}_\lambda; \tag{30a}$$

$$\Big[\boldsymbol{u} + \nabla_x p_* - \boldsymbol{g}(\theta(\lambda))\Big]\, \frac{\partial f}{\partial \lambda} = 0, \qquad (\boldsymbol{x}, t, \lambda) \in Q \times \mathbb{R}_\lambda; \tag{30b}$$

$$\operatorname{div}_x \boldsymbol{u} = 0, \qquad (\boldsymbol{x}, t) \in Q \tag{30c}$$

and the initial conditions

$$f(\boldsymbol{x}, 0, \lambda) = f_0(\boldsymbol{x}, \lambda), \qquad (\boldsymbol{x}, \lambda) \in \Omega \times \mathbb{R}_\lambda. \tag{30d}$$

The parameterized measures $\nu_{x,t}$ and $\sigma_{x,t}$ in Secs. 1 and 3 of the formulation of problem K in the course of the limit transition as $\varepsilon_k \to 0$ follow from the limit relations (19) and (20) as the Young measures associated with the sequences $\{e_{\varepsilon_k}\}$ and $\{e_{\varepsilon_k}, \nabla_x H(e_{\varepsilon_k})\}$, respectively.

Equation (30c) is satisfied almost everywhere in Q. The initial conditions (30d) are understood in the sense of a weak trace. Equations (30a) and (30b) are understood in the sense of distributions and, with allowance for Eqs. (30c) and (30d), can be presented as a system of integral equalities

$$\int\limits_{Q \times \mathbb{R}_\lambda} \Big(\partial_t \zeta + I_{\lambda \geq 0}\, \boldsymbol{u} \cdot \nabla_x \zeta + \theta'(\lambda)\, \Delta_x \zeta\Big) f(\boldsymbol{x}, t, \lambda)\, d\boldsymbol{x}\, dt\, d\lambda + \int\limits_{Q \times \mathbb{R}_\lambda} \partial_\lambda \zeta\, dM$$
$$+ \int\limits_Q \Big(\int\limits_{\mathbb{R}_\lambda} \partial_\lambda \zeta\, d_\lambda \chi(\boldsymbol{x}, t, \lambda)\Big)\, d\boldsymbol{x}\, dt + \int\limits_{\Omega \times \mathbb{R}_\lambda} \zeta(\boldsymbol{x}, 0, \lambda) f_0(\boldsymbol{x}, \lambda)\, d\boldsymbol{x}\, d\lambda = 0; \tag{31}$$

$$\int\limits_{Q \times \mathbb{R}_\lambda} \Big\{\Big[\boldsymbol{u} + \nabla_x p_* - \boldsymbol{g}(\theta(\lambda))\Big] \cdot \partial_\lambda \boldsymbol{\eta} - \partial_\lambda \boldsymbol{g}(\theta(\lambda)) \cdot \boldsymbol{\eta}\Big\} f(\boldsymbol{x}, t, \lambda)\, d\boldsymbol{x}\, dt\, d\lambda = 0. \tag{32}$$

In Eqs. (31) and (32), $\zeta(\boldsymbol{x}, t, \lambda)$ is an arbitrary smooth test function 1-periodic with respect to $\boldsymbol{x}$, which vanishes in the neighborhood of the plane $\{t = T\}$ for rather high values of λ; $\boldsymbol{\eta}(\boldsymbol{x}, t, \lambda)$ is an arbitrary smooth test vector function 1-periodic with respect to $\boldsymbol{x}$, which vanishes for rather high values of λ.

5. Solvability of the Darcy–Stefan Problem. Kinetic equations of the form (30a) with involved functions and measures satisfying conditions similar to requirements to the formulation of problem K in Secs. 1–4 were studied in detail in [13, 15]. For Eq. (30a), these works yield two statements, which are the key statements in the proof of solvability of the Darcy–Stefan problem.

Statement 3. *For smooth and convex functions Φ on the segment $[0,1]$, there exists the Borel measure $\Lambda_\Phi \in C(Q\times\mathbb{R}_\lambda)^*$ with a support belonging to the band $-c_0 \le \lambda \le c_0$ such that the following renormalized inequality is satisfied:*

$$\int\limits_{Q\times\mathbb{R}_\lambda} \Phi(f)\Big\{\partial_t\zeta + I_{\lambda\ge 0}\,\boldsymbol{u}\cdot\nabla_x\zeta + \theta'(\lambda)\,\Delta_x\zeta\Big\}\,d\boldsymbol{x}\,dt\,d\lambda$$

$$+\int\limits_{\Omega\times\mathbb{R}_\lambda} \Phi(f_0)\zeta(\boldsymbol{x},0,\lambda)\,d\boldsymbol{x}\,d\lambda - \int\limits_{Q\times\mathbb{R}_\lambda} \partial_\lambda\zeta\,d\Lambda_\Phi(\boldsymbol{x},t,\lambda) \le 0. \tag{33}$$

Here $\zeta(\boldsymbol{x},t,\lambda)$ is an arbitrary non-negative smooth test function 1-periodic with respect to $\boldsymbol{x}$, which vanishes in the neighborhood of the plane $\{t=T\}$ for rather high values of $|\lambda|$.

The proof is similar to the proof of Theorem 6 in [13].

With allowance for the initial condition $f(\boldsymbol{x},0,t) = f_0(\boldsymbol{x},\lambda)$, inequality (33) in the sense of distributions is equivalent to the renormalized inequality

$$\frac{\partial\Phi(f)}{\partial t} + I_{\lambda\ge 0}\boldsymbol{u}\cdot\nabla_x\Phi(f) - \theta'(\lambda)\,\Delta_x\Phi(f) \ge -\frac{\partial\Lambda_\Phi}{\partial\lambda}, \qquad (\boldsymbol{x},t,\lambda)\in Q\times\mathbb{R}_\lambda.$$

Statement 4. *The following items are valid.*

1. *Solutions of problem K satisfy the equation*

$$f(\boldsymbol{x},t,\lambda)(1-f(\boldsymbol{x},t,\lambda)) = 0 \quad \textit{almost everywhere in} \quad \mathbb{R}^d_x\times(0,T)\times\mathbb{R}_\lambda,$$

i.e., the distribution function f takes only the values 0 and 1.

2. *The distribution function $f(\boldsymbol{x},t,\lambda)$ has the structure of the Heaviside function of the variable λ, i.e., there exists a function $e_* = e_*(\boldsymbol{x},t)$ such that $e_*\in L^\infty(0,T;L^\infty)$, $-c_0\le e_*\le c_0$ almost everywhere in $\mathbb{R}^d_x\times(0,T)$, and*

$$f(\boldsymbol{x},t,\lambda) = I_{\lambda\ge e_*(x,t)}. \tag{34}$$

The proof of item 1 is similar to the proof in [13, Sec. 5] and [15] and is based on choosing the test functions in (33) in the form $\Phi(f) = f(f-1)$ and $\zeta(\boldsymbol{x},t,\lambda) = \zeta_1(\lambda)\zeta_2(t)$, where ζ_1 is non-negative and equal to unity on the segment $[-c_0,c_0]$, and ζ_2 is non-negative, vanishes at $t=T$, and rigorously decreases at $t\in[0,T)$.

Presentation (34) follows from item 1 of Statement 4 and also from the fact that f satisfies requirements (1) of the formulation of problem K.

Let us return to the proof of Theorem 1. By virtue of Eq. (21), presentation (34) implies that the Young measure $\nu_{x,t}$ is a parameterized Dirac measure concentrated at the point $\lambda = e_*(\boldsymbol{x},t)$. According to the theory of Young measures (Theorem 2.31 in [10, Chapter 3]), this means that e_* coincides almost everywhere in $\mathbb{R}^d_x\times(0,T)$ with e [limit function in Eq. (9)] and

$$e_\varepsilon \to e \quad \text{strongly in} \quad L^2_{\rm loc}(\mathbb{R}^d_x\times(0,T)). \tag{35}$$

Thus, strong precompactness of the family of approximate solutions $\{e_\varepsilon, p_{*\varepsilon}, \boldsymbol{u}_\varepsilon\}$ is proved.

Substituting presentation (34) (in which, as was mentioned previously, $e_* = e$ almost everywhere in Q) and test functions of the form $\zeta(\boldsymbol{x},t,\lambda) := \varphi(\lambda)\zeta_0(\boldsymbol{x},t)$ and $\boldsymbol{\eta}(\boldsymbol{x},t,\lambda) = \eta_1(\lambda)\boldsymbol{\eta}_2(\boldsymbol{x},t)$ into the integral inequalities (30a) and (30b) and integrating with respect to λ with allowance for the non-negative value of the measure M and the structure of the measure $d_\lambda\chi$, we conclude that the pair of functions (e,p_*) obtained by the limit transitions (35) and (24) is the entropy solution of the Darcy–Stefan problem in the sense of the definition of the entropy solution. It should be also noted that $\boldsymbol{u} = -\nabla_x p_* + \boldsymbol{g}(\theta(e))$ almost everywhere in $\mathbb{R}^d_x\times(0,T)$ by virtue of equalities (30b) and (34). In particular, it follows from here that the velocity vector $\boldsymbol{v}(\boldsymbol{x},t)$ in the Darcy–Stefan problem is reconstructed by the formula $\boldsymbol{v} = I_{e\ge 0}\boldsymbol{u}$. Theorem 1 is proved.

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